OPERATOR SPLITTING FOR CONVECTION-DOMINATED NONLINEAR PARTIAL DIFFERENTIAL EQUATIONS

HELGE HOLDEN

Department of Mathematical Sciences,
Norwegian University of Science and Technology,
N–7491 Trondheim, Norway.
e-mail: holden@math.ntnu.no

KENNETH HVISTENDAHL KARLSEN

Department of Mathematics,
University of Bergen,
Johs. Brunsgt. 12,
N–5008 Bergen, Norway.
e-mail: kennethk@mi.uib.no

KNUT–ANDREAS LIE

Department of Informatics,
University of Oslo,
P.O. Box 1080 Blindern,
N–0316 Oslo, Norway.
e-mail: kalie@ifi.uio.no

AND

NILS HENRIK RISEBRO

Department of Mathematics,
University of Oslo,
P.O. Box 1053 Blindern,
N–0316 Oslo, Norway.
e-mail: nilshr@math.uio.no

1. Introduction

We describe an efficient solution strategy for nonlinear systems of partial differential equations of the form

$$U_t + \sum_i F_i(U)_{x_i} = \sum_{i,j} D_{ij}(U)_{x_i x_j} + G(U), \quad U|_{t=0} = U_0. \tag{1}$$

We explicitly allow for degeneracy of the viscous term in the sense that we only require $\sum_{i,j} D'_{ij}(u)\xi_i\xi_j \geq 0$. The solution strategy is based on operator splitting where an abstractly defined Cauchy problem $U_t + \mathcal{A}(U) = 0$, is split into simpler problems $V_t^l + \mathcal{A}^l(V^l) = 0$, by writing $\mathcal{A} = \mathcal{A}^1 + \cdots + \mathcal{A}^\ell$. If the solution of subproblem l is written $V^l(t) = \mathcal{S}_t^l V_0^l$ then the idea is that an approximate solution of the original problem reads

$$U(n\Delta t) \approx U^n = \left[\mathcal{S}_{\Delta t}^\ell \circ \cdots \circ \mathcal{S}_{\Delta t}^1\right]^n U^0, \qquad t = n\Delta t.$$

From a numerical point of view, the idea behind operator splitting is to combine efficient and accurate numerical methods for each of the subproblems to build an overall solution strategy. This allows for a solver with great modularity, where each component can easily be replaced. Operator splitting has therefore been a common strategy over the last decades.

Operator splitting may introduce new numerical difficulties and lead to the computation of nonphysical numerical artifacts. Lately, much of the development of numerical methods within the scientific computing community therefore aims at incorporating as many of the physical effects as possible into a single method. On the other hand, there is an increasing influx of modern software tools such as object-orientation into scientific computing. In object-orientation one often seeks to identify simple and generic objects that can be combined to solve complicated problems. The generic objects can then be implemented, tested and verified independently and reused in other settings. In this sense, operator splitting is an object-oriented approach to solving (1). However, equipped with a large toolbox of small parts that can be fitted together independently (like Lego pieces), the natural question is whether the new creation will work as planned?

This question is normally answered in the form of a mathematical convergence theory; an abstract and general convergence theory that includes a large number of previous specific splitting methods for (1) is presented in (Holden et al., 1999).

Theoretical convergence is not sufficient from a practical point of view; accuracy and efficiency for given discretization parameters are often more important. We therefore present several splitting methods developed especially to capture sharp gradient variations in the solutions of (1). The distinct feature of our approach is the use of a large-time-step front-tracking method (Holden and Holden, 1998; Risebro, 1993; Bressan, 1992) to solve one-dimensional hyperbolic subproblems of the form $U_t + F(U)_x = 0$. The integrated method is unconditionally stable and delivers more than the standard resolution with surprisingly high efficiency. Our methods have been applied to specific problems arising when simulating flow in porous media, gas dynamics, shallow water waves, glaciers, traffic flow, and sedimentation.

2. Operator Splitting Methods

In the following we present some examples of efficient operator splitting methods based on front tracking, which we introduce first.

Front Tracking. The term 'front-tracking' is applied to a wide variety of methods with the common feature that they seek to perform explicit tracking of discontinuities in hyperbolic solutions. Our method originates from an idea by Dafermos. Consider the conservation law

$$U_t + F(U)_x = 0, \qquad U(x,0) = U_0(x). \tag{2}$$

Making the usual piecewise constant approximation to the initial data, the Cauchy problem is converted into a sequence of Riemann problems. The essence of the method is to approximate the solution of each Riemann problem by a step function, i.e., by a set of constant states separated by space-time rays of discontinuity (*fronts*), and track the discontinuities explicitly. Each time two or more fronts collide, they define a new Riemann problem which is approximated by piecewise constants, and so on.

For scalar equations, the approximation of Riemann problems is typically achieved by making a piecewise linear approximation to the flux function. In the systems case, one retains shocks and constants and approximates rarefaction waves. The front tracking method is unconditionally stable and has first order convergence with respect to the approximation of the initial data and the Riemann problems, see (Holden and Holden, 1998).

Figure 1 shows the front tracking approximation for a scalar problem (2) with flux function $f(u) = 2u^2(1-u^2)$ and initial data $u_0(x) = \sin(\pi x)\chi_{[-1,1]}(x)$.

The front tracking method is easily extended to quasilinear equations with variable coefficients $u_t + V(x,t)f(u)_x = 0$ by introducing e.g., a polynomial approximation to the velocity field (Lie, 1999).

Example 1. The natural extension of front tracking to multidimensions is by dimensional splitting on a Cartesian grid (Holden and Risebro, 1993). Since front tracking is unconditionally stable, the step size in the splitting is not limited by a stability condition. Instead, the limiting factor lies in the two error mechanisms; temporal splitting errors that increase with Δt

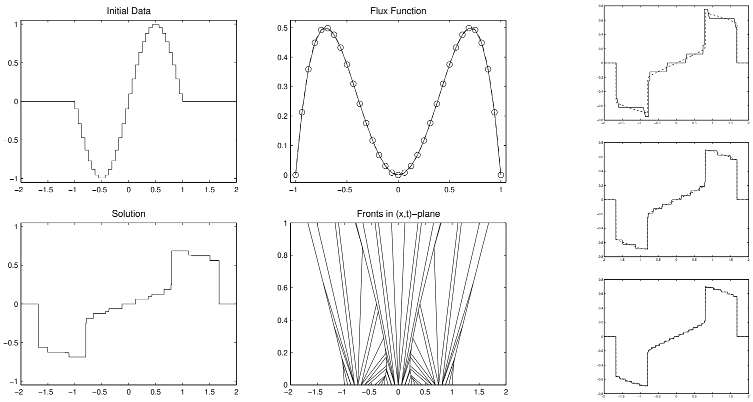

Figure 1. (Left) Front tracking solution for a scalar 1-D problem with $n = 64$ intervals in approximation of u_0 and $m = 32$ intervals for f. (Right) Refinement of flux approximation: $m = 16$, 32, and 64 from top to bottom for $n = 2048$.

and spatial errors from the projections that decrease with Δt, see Figure 2. The theoretical convergence order is one half (Lie, Haugse, and Karlsen, 1998) and is observed for *linear* problems, see (Lie, 1999). For nonlinear problems, convergence of order one is observed (Lie, Haugse, and Karlsen, 1998; Lie, 1999) due to nonlinear self-sharpening effects that counteract the smearing from the projections.

For many problems one can therefore use surprisingly large CFL numbers. Figure 3 illustrates this for Burgers' equation with a variable velocity field $V = (\cos(\pi(y+t)), \sin(\pi(x+t)))$. Although not all details are resolved accurately, the qualitative representation is quite good. Extensive numerical experiments indicate that the method gives best performance for CFL numbers around 10-20. The large-step ability of the splitting method makes it very efficient, typically a factor 40-50 faster than standard high-resolution method when comparing runtime versus accuracy, see (Lie, Haugse, and Karlsen, 1998; Lie, 1999).

Example 2. The next example illustrates the use of dimensional splitting to solve multidimensional systems by front tracking, see (Holden, Lie, and Risebro, 1999) for more examples. Here we consider the Euler equations of gas dynamics describing an ideal, polytropic gas with gas constant 1.4. Our test case is a cylindrical Riemann problem between two horizontal walls, see Figure 4. The initial Riemann problem leads to an inward moving rarefaction and a strong outward moving shock followed by a contact. The first shock reflects at the lower wall. At time $t = 0.2$ the reflected shock has passed through the contact and into the low-density region behind, where the shock speed increases. The rarefaction wave implodes on the cylinder center and produces an outward moving shock.

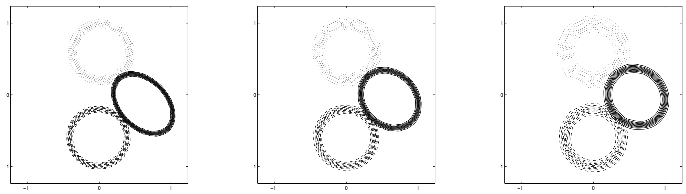

Figure 2. Solid body rotation around the origin ($u_t + yu_x - xu_y = 0$) of a cylinder of radius 0.4 centred at (0,0.6). Solutions at time $t = \pi/2$ (solid), π (dashed), and 2π (dotted) computed with 16 (left), 32, and 64 splitting steps (right) and $\Delta x = 0.025$. The corresponding CFL numbers are 20, 10, and 5.

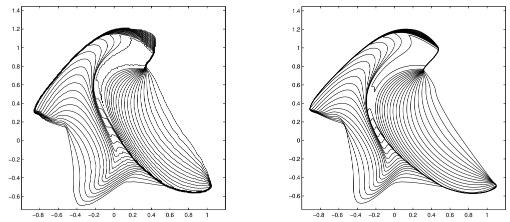

Figure 3. Solution of $u_t + \nabla \cdot (u^2 V) = 0$ at time $t = 1.0$ computed with 8 splitting steps and $\Delta x = 0.01$ (left) corresponding to CFL number $\nu \approx 25$ compared with a fine grid solution (right). The initial data equals one inside a square with sides lengths 0.5 centered at the origin and zero outside.

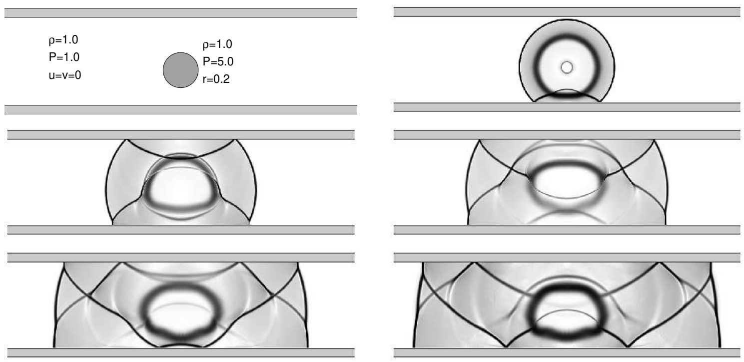

Figure 4. Evolution of a high pressure cylinder between two infinite walls; initial setup and emulated Schlieren images at times $t = 0.2$, 0.4, 0.6, 0.8, and 1.0.

For systems, the generation of small-scale oscillations prevents the use of very large time steps and the method performs best at CFL numbers moderately above unity (typically 1-4), see (Holden, Lie, and Risebro, 1999).

Example 3. Operator splitting methods are often applied to solve

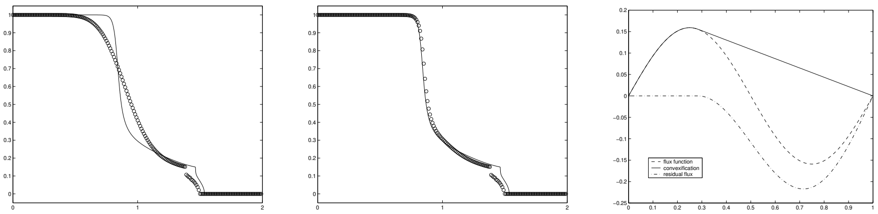

Figure 5. Degenerate convection-diffusion equation with $f(u) = \sin(2\pi u)/2\pi$, $D'(u) = \chi_{[0.15,\infty)}(u)$, and $u_0(x) = \chi_{(-\infty,1]}(u)$. (Left) Operator splitting solution at time $t = 0.5$ computed with one step. (Middle) Corrected operator splitting. (Right) Convexification of flux function and the corresponding residual.

convection-diffusion equations on the form $u_t + f(u)_x = D(u)_{xx}$. A straightforward splitting decomposes the equation into a hyperbolic part $u_t + f(u)_x = 0$ and a diffusive part $u_t = D(u)_{xx}$. For convection dominated problems this may lead to significant smearing of sharp gradients unless the splitting step is chosen very small. This viscous splitting error comes from the entropy loss imposed by local linearization of f during the hyperbolic step (Karlsen and Risebro, 2000). The entropy loss can be identified in the form of local residual fluxes that ensure the correct amount of self sharpening in shock layers. By including these residuals in the splitting, e.g., in the form of a modified diffusive step $u_t + f_{\text{res}}(u; x)_x = D(u)_{xx}$, one obtains a robust splitting strategy, called *corrected operator splitting*, see (Karlsen and Risebro, 2000; Espedal and Karlsen, 2000). This is illustrated in Figure 5, where the shock layer is resolved correctly by the modified splitting. However, the interface between the hyperbolic and the parabolic region, which depends on $f(u)$ and $D'(u)$, is not resolved by one splitting step. An overview of general splitting methods are given in (Espedal and Karlsen, 2000).

Example 4. The same splitting strategy can be applied in multidimensions, either using first dimensional splitting and then viscous splitting for the one-dimensional problems, or first viscous splitting and the dimensional splitting for the multidimensional hyperbolic problem. We adopt the latter strategy and consider two-phase flow in a reservoir. This process is governed by an elliptic pressure equation $\nabla(\lambda \nabla p) = 0$ that is coupled to a saturation equation $s_t + V \cdot \nabla f(s) = \varepsilon \nabla(K d(s) \nabla s)$ through the Darcy velocity $V = -\lambda \nabla p$. The system is typically solved by a sequential splitting; first solve for p and compute V, then V is held fixed as s is updated, and so on. Figure 6 shows the quarter five-spot test case with a heterogeneous permeability field K, i.e., the first quadrant of a periodic well configuration with a water injection well in the origin and production wells at $(\pm 1, \pm 1)$.

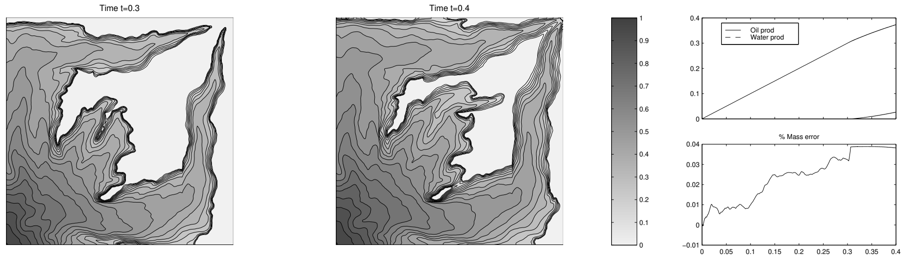

Figure 6. Operator splitting solution with CFL number 4.0 and 16 sequential steps. The flux function is $f(s) = s^2/(s^2 + 0.125(1-s)^2)$, the capillary diffusion function $d(s) = 4s(1-s)$, and $\varepsilon = 0.005$. Water breaks through in the production well after the injection of 0.306 pore volumes of water.

References

Bressan A (1992). Global solutions of systems of conservation laws by wave-front tracking. *J. Math. Analysis and App.*, **170**: 414–432.

Espedal M S and Karlsen K H. Numerical solution of reservoir flow models based on large time step operator splitting algorithms. To appear in A. Fasano and H. van Duijn, editors, *Filtration in Porous Media and Industrial Applications*, Lecture Notes in Mathematics, Springer Verlag.

Holden H and Holden L (1998). On scalar conservation laws in one-dimension. In *Ideas and Methods in Mathematics and Physics*, pages 480–509. Cambridge University Press.

Holden H, Karlsen K H, Lie K-A, and Risebro N H (1999). Operator splitting for nonlinear partial differential equations: An L^1 convergence theory. In preparation.

Holden H, Lie K-A, Risebro N H (1999). An unconditionally stable method for the Euler equations. *J. Comput. Phys.*, **150(1)**: 76–96.

Holden H and Risebro N H (1993). A method of fractional steps for scalar conservation laws without the CFL condition. *Math. Comp.*, **60(201)**: 221–232.

Karlsen K H and Risebro N H (2000). Corrected operator splitting for nonlinear parabolic equations. *SIAM J. Numer. Anal.*, **37(3)**: 980–1003.

Lie K-A, Haugse V, and Karlsen K H (1998). Dimensional splitting with front tracking and adaptive local grid refinement. *Numer. Methods Partial Differential Equations*, **14(5)**: 627–648.

Lie K A (1999). A dimensional splitting method for quasilinear hyperbolic equations with variable coefficients. *BIT*, **39(4)**: 683–700.

Risebro N H (1993). A front-tracking alternative to the random choice method. *Proc. Amer. Math. Soc.*, **117(4)**: 1125–1139.